Implicit Surface Fitting by Regularized Regression and Compactly Supported Radial Basis Functions

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Abstract
We describe a direct way of making use of surface normal vectors at sample points in the problem of implicit surface fitting with compactly supported radial basis functions. The normal vectors are incorporated in a regularized regression problem that leads to a $n \times n$ positive definite linear system given n surface point/normal pairs $(\mathbf{x}_i, \mathbf{n}_i)$. Compared with the widely used heuristic methods, our method avoids of introducing manufactured off-surface points and can fit much larger datasets effectively. We demonstrate its robust performance on several datasets.
Keywords: *Implicit Surface Fitting, Radial Basis Function, Regularized Regression*

1 Introduction

Reconstructing a surface from sample points is a powerful 3D modeling method in CAD, reverse engineering, cultural heritage protection and other applications. Recent improvements of shape acquisition techniques such as laser scanners, optical triangulation systems and mechanical probing methods have made it easier to digitize a world object into large point sets. However, these point sets are often unorganized, noisy, contain holes and make it a challenging problem to infer a surface.

Reconstruction approaches can be roughly classified as Delaunay based methods and implicit surface methods [9]. Among various implicit surface methods, the Radial Basis Functions (RBF) approach has shown successful at reconstructing water-tight surfaces of arbitrary topology from scattered point samples [2,4,5,6,8,10,12,13]. It approximates the input surface as the zero level-set of a scalar 3D function $f(\mathbf{x})$ which is expressed as a weighted sum of radial basis functions. Given a set of centers of basis functions, a set of constraint points and a type of radial basis function, the set of weights can be determined by a linear system of equations.

Presently, there are two kinds of radial basis functions. Functions with global support can produce excellent reconstruction results, but lead to a dense linear system [2,10,12]. This approach is therefore tractable only for small data sets. The viable solution to the computational problem so far is the FastRBF algorithm that uses the Fast Multipole Method (FMM) [2].Unfortunately, it is difficult to implement and exists only in the proprietary FastRBF package up to date. Another means of overcoming the computational problem is to use compactly supported radial basis functions (CSRBF) which lead to a sparse linear system and fast algorithms. Although it is sensitive to the quality of input data and lack of extrapolation across large holes, it is easy to implement and can fit large point sets effectively [4,5,6,8,13].

In order to avoid the trivial solution that f is zero everywhere, two main options exist. The first approach is to introduce some off-surface points, where the function is constrained to be non-zero. Usually, at each surface point $\mathbf{x}_i$, a pair of off-surface points $\mathbf{x}_i \pm d\mathbf{n}_i$ are manufactured along its unit length surface normal vector $\mathbf{n}_i$ with some heuristically chosen d. The function value at the off-surface point is taken to be the distance to the nearest surface point. However, these off-surface points may contradict one another and some heuristics are employed to ensure that off-surface points produce a distance field consistent with the surface data. Samozino et al [8] select the off-surface points among the vertices of the Voronoi diagram of the input data points. At each off-surface point, the radius of the polar ball approximates its distance to the sampled shape, and the sign of the value is determined by classifying the pole as inside or outside. This technique needs a user-defined budget of centers and its efficiency and scalability need to be improved. The off-surface points can also be generated by some space structures and given values by a local approximation of subsets of nearby points [6,7]. To avoid the use of off-surface points, Walder et al. [13] propose a direct way of using the normal vectors in the regularised interpolation problem. Unfortunately this method leads to a $4n \times 4n$ linear system of equations given n surface point/normal pairs $(\mathbf{x}_i, \mathbf{n}_i)$, which is intractable for large data sets.

Based on Walder's method [13,14], we propose a novel approach to incorporate the normal vectors in a regularized regression problem, which leads to a $n \times n$ positive definite linear system given n surface point/normal pairs. Compared with the widely used heuristic methods and Walder's method, our approach avoids of introducing manufactured off-surface points and can fit much larger datasets effectively. The formal derivation is given in Section 2 and details of implementation are given in Section 3. Section 4 presents a series of tests and comparisons, and some future works are discussed in Section 5.

2 Implicit Surface Fitting by Regularized Regression

In implicit surface modeling, we want to fit a function f to a given set of 3D points $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ where their estimated signed distances are $\{y_1, \cdots, y_n\}$. The usual approach based on regularised interpolation is to find the minimizer f of

$$E(f) = \| f \|_{\mathrm{H}}^2 + C \sum_{i=1}^{n} \left(f(\mathbf{x}_i) - \mathrm{y}_i \right)^2$$

We assume that surface normals at points $\{\mathbf{x}_1,\cdots,\mathbf{x}_n\}$ are given or can be estimated from nearest neighbor information. Since the normal direction of the implicit surface is given by the gradient of the embedding function f, the normal vectors $\{\mathbf{n}_1,\cdots,\mathbf{n}_n\}$ at the given points can be directly incorporated into the optimization problem by

$$E(f)=\|f\|_{\mathrm{H}}^2+C_1\sum_{i=1}^{n}\left(f(\mathbf{x}_i)-\mathrm{y}_i\right)^2-C_2\sum_{i=1}^{n}\left(\mathbf{n}_i^{\mathrm{t}}\nabla f(\mathbf{x}_i)\right) \tag{1}$$

where $C_1>0$ and $C_2>0$ are parameters to weigh between fitness to the data points/normals and smoothness of the surface. The last term directly measures the alignment of ∇f with the normal direction. The norm $\|f\|_{\mathrm{H}}$ is a regulariser that takes on larger values for less smooth functions. H is a reproducing kernel Hilbert space (RKHS) with kernel function $k(\cdot,\cdot)$ and there is a reproduction property

$$f(\mathbf{x})=\left\langle f(\mathbf{y}),k(\mathbf{x},\mathbf{y})\right\rangle_{\mathrm{H}}$$

A necessary condition of optimality is

$$\frac{\partial E(f)}{\partial f}=\left(0\in \mathrm{H}\right)$$

$$\frac{\partial E(f)}{\partial f}=\left(0\in \mathrm{H}\right) \tag{2}$$

With

$$\frac{\partial}{\partial f}\left(\mathbf{n}_i^{\mathrm{t}}\nabla f(\mathbf{x}_i)\right)=\frac{\partial}{\partial f}\lim_{\varepsilon\to 0}\sum_{l=1}^{3}\left([\mathbf{n}_i]_l\left(f(\mathbf{x}_i+\varepsilon\mathbf{e}_l)-f(\mathbf{x}_i)\right)/\varepsilon\right)$$

$$=\lim_{\varepsilon\to 0}\sum_{l=1}^{3}\left([\mathbf{n}_i]_l\left(k(\mathbf{x}_i+\varepsilon\mathbf{e}_l,\mathbf{y})-k(\mathbf{x}_i,\mathbf{y})\right)/\varepsilon\right)$$

$$=\mathbf{n}_i^{\mathrm{t}}\nabla k(\mathbf{x}_i,\mathbf{y})$$

Equation (2) leads to

$$\frac{\partial E(f)}{\partial f}=\left(0\in \mathrm{H}\right)=2f+2C_1\sum_{i=1}^{n}\left(f(\mathbf{x}_i)-\mathrm{y}_i\right)k(\mathbf{x}_i,\mathbf{y})-C_2\sum_{i=1}^{n}\left(\mathbf{n}_i^{\mathrm{t}}\nabla k(\mathbf{x}_i,\mathbf{y})\right)$$

$$f(\mathbf{x}) = -C_1\sum_{i=1}^{n}(f(\mathbf{x}_i)-\mathrm{y}_i)k(\mathbf{x}_i,\mathbf{x}) + \frac{C_2}{2}\sum_{i=1}^{n}\left(\mathbf{n}_i^{\mathrm{t}}\nabla k(\mathbf{x}_i,\mathbf{x})\right)$$

$$= \sum_{i=1}^{n}\alpha_i k(\mathbf{x}_i,\mathbf{x}) + \frac{C_2}{2}\sum_{i=1}^{n}\left(\mathbf{n}_i^{\mathrm{t}}\nabla k(\mathbf{x}_i,\mathbf{x})\right)$$

where

$$\alpha_i = -C_1(f(\mathbf{x}_i)-\mathrm{y}_i)$$

So we have to solve for n coefficients α_i to obtain the final solution

$$f(\mathbf{x}) = \sum_{i=1}^{n}\alpha_i k(\mathbf{x}_i,\mathbf{x}) + \frac{C_2}{2}\sum_{i=1}^{n}\left(\mathbf{n}_i^{\mathrm{t}}\nabla k(\mathbf{x}_i,\mathbf{x})\right) \quad (3)$$

Defining element-wise matrices:

$$[\mathbf{f}]_i = f(\mathbf{x}_i),\ [\boldsymbol{\alpha}]_i = \alpha_i,\ [\mathbf{y}]_i = y_i,\ [\mathbf{K}]_{i,j} = k(\mathbf{x}_j,\mathbf{x}_i),$$

$$[\nabla\mathbf{K}]_i = \left[(\nabla k(\mathbf{x}_1,\mathbf{x}_i))^{\mathrm{t}},\cdots,(\nabla k(\mathbf{x}_n,\mathbf{x}_i))^{\mathrm{t}}\right]$$

$$\mathbf{n} = \left[\mathbf{n}_1^{\mathrm{t}},\cdots,\mathbf{n}_n^{\mathrm{t}}\right]^{\mathrm{t}}$$

We get $\boldsymbol{\alpha} = -C_1(\mathbf{f}-\mathbf{y})$ and Equation (3) can be rewritten as

$$-\frac{1}{C_1}\boldsymbol{\alpha} + \mathbf{y} = \mathbf{K}\boldsymbol{\alpha} + \frac{C_2}{2}\nabla\mathbf{K}\mathbf{n}$$

which yields

$$\left(\mathbf{K}+\mathbf{I}/C_1\right)\boldsymbol{\alpha} = \mathbf{y} - \frac{C_2}{2}\nabla\mathbf{K}\mathbf{n}$$

Thus given n surface point/normal pairs $(\mathbf{x}_i,\mathbf{n}_i)$, the coefficients $\boldsymbol{\alpha}$ of the solution can be obtained by solving a positive definite linear system:

$$\left(\mathbf{K}+\mathbf{I}/C_1\right)\boldsymbol{\alpha} = -\frac{C_2}{2}\nabla\mathbf{K}\mathbf{n} \quad (4)$$

3 Implementation

We now describe the implementation details in the process of implicit surface fitting.

3.1 Choice of RBF

The compactly supported radial basis function first appeared in the literature in the mid 1990s, and most popular classes of CSRBFs are those introduced by Wu [16] and Wendland [15]. We use the $C^2(\mathrm{R}^3)$ Wu function as a kernel in our implementation.

$$k(\mathbf{x}_i,\mathbf{x})=k(r)=(1-\frac{r}{\delta})_+^4(4+16\frac{r}{\delta}+12\frac{r^2}{\delta^2}+3\frac{r^3}{\delta^3})$$

where

$$r=\|\mathbf{x}_i-\mathbf{x}\|=\left[(x_i-x)^2+(y_i-y)^2+(z_i-z)^2\right]^{1/2},$$

$\delta>0$ is the size of support for CSRBFs, and the symbol $+$ means $(\mathrm{x})_+=\mathrm{x}$ if $\mathrm{x}>0$ and $(\mathrm{x})_+=0$ otherwise. Its partial derivatives can be obtained as follows:

$$\frac{\partial k}{\partial x}=\frac{\partial k}{\partial r}\frac{\partial r}{\partial x},\qquad \frac{\partial k}{\partial y}=\frac{\partial k}{\partial r}\frac{\partial r}{\partial y},\qquad \frac{\partial k}{\partial z}=\frac{\partial k}{\partial r}\frac{\partial r}{\partial z}$$

where

$$\frac{\partial k}{\partial r}=\frac{1}{\delta}\left[(1-\frac{r}{\delta})_+^4(16+24\frac{r}{\delta}+9\frac{r^2}{\delta^2})-4(1-\frac{r}{\delta})_+^3(4+16\frac{r}{\delta}+12\frac{r^2}{\delta^2}+3\frac{r^3}{\delta^3})\right],$$

$$\frac{\partial r}{\partial x}=\frac{x-x_i}{r},\qquad \frac{\partial r}{\partial y}=\frac{y-y_i}{r},\qquad \frac{\partial r}{\partial z}=\frac{z-z_i}{r}$$

3.2 Parameter Selection

The support size δ is estimated as recommended by [6]. We use an octree-based data structure of the input points with each leaf cell containing no more than eight points, and delete the leaf cells containing no points. We set δ equal to the average diagonal length of the leaf cells.

In Equation (4), we set $\boldsymbol{\beta}=2\boldsymbol{\alpha}/C_2$ and get the following linear system:

$$\left(\mathbf{K}+\mathbf{I}/C_1\right)\boldsymbol{\beta}=-\nabla\mathbf{K}\mathbf{n} \tag{5}$$

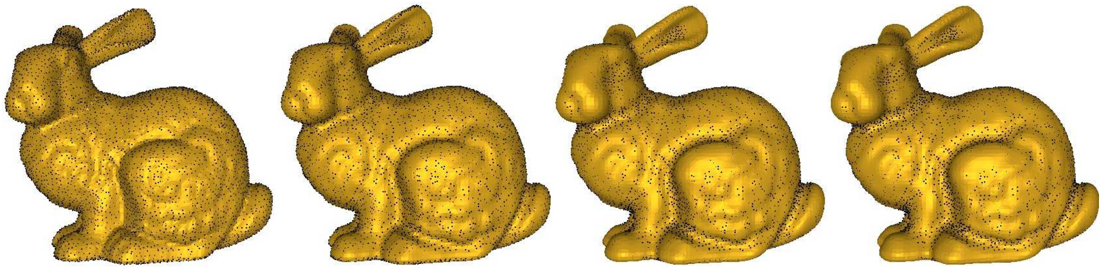
The function f can be written as

$$f(\mathbf{x})=\frac{C_2}{2}f'(\mathbf{x})=\frac{C_2}{2}\left[\sum_{i=1}^{n}\beta_i k(\mathbf{x}_i,\mathbf{x})+\sum_{i=1}^{n}\left(\mathbf{n}_i^t\nabla k(\mathbf{x}_i,\mathbf{x})\right)\right] \tag{6}$$

Clearly, the parameter C_2 can be estimated by some heuristics after we solve the linear system. We generate an additional point $\mathbf{x}_i^+ = \mathbf{x}_i + 0.5\delta\mathbf{n}_i$ by displacing $\mathbf{x}_i$ along its normal $\mathbf{n}_1$. We discard the point and take another point as $\mathbf{x}_i$ if there exists a given surface point at a distance of less than 0.5δ. The parameter C_2 can be taken as

$$C_2 = \delta / \left[f'(\mathbf{x}_i^+) - f'(\mathbf{x}_i) \right]$$

The parameter C_1 affects the fitness to the data points and smoothness of the surface. We can get an exact fit to the raw data with $\rho = 1/C_1 = 0$. A larger value of ρ leads to increased smoothing. It can be estimated by the user or chosen automatically using k-fold cross validation. Fig.1 illustrates the smoothing effects of varying ρ.



(a) (b) (c) (d)

Figure 1: Various values of ρ lead to various amounts of smoothing effects. (a) Exact fit to the sample points (ρ =0), (b) ρ =10, (c) ρ =100, (d) ρ =1000. The sample points are represented as black dots.

3.3 Solver and Isosurface Extraction

We used the TAUCS library [11] to solve the linear system and polygonized the implicit surface by Bloomenthal's method [1].

4 Results

We processed a number of 3D point sets obtained from the AIM@SHAPE repository [17]. The results are presented in Fig.2 and timings are given in Table 1.

(a) (b) (c) (d)

Figure 2: Examples of implicit surface fitting. From left to right, the Chinese dragon, the Armadillo, the Hand_olivier and the Sforza model.

Table 1. Results with a 3.00 GHz Pentiumn 4 processor and 1 GB Memory, for various 3D data sets. Column two is the number of sample points. Column three is the estimated support size of CSRBF. Column four and five are time in seconds for building and solving the linear system , respectively. The final column is the total fitting time in seconds.

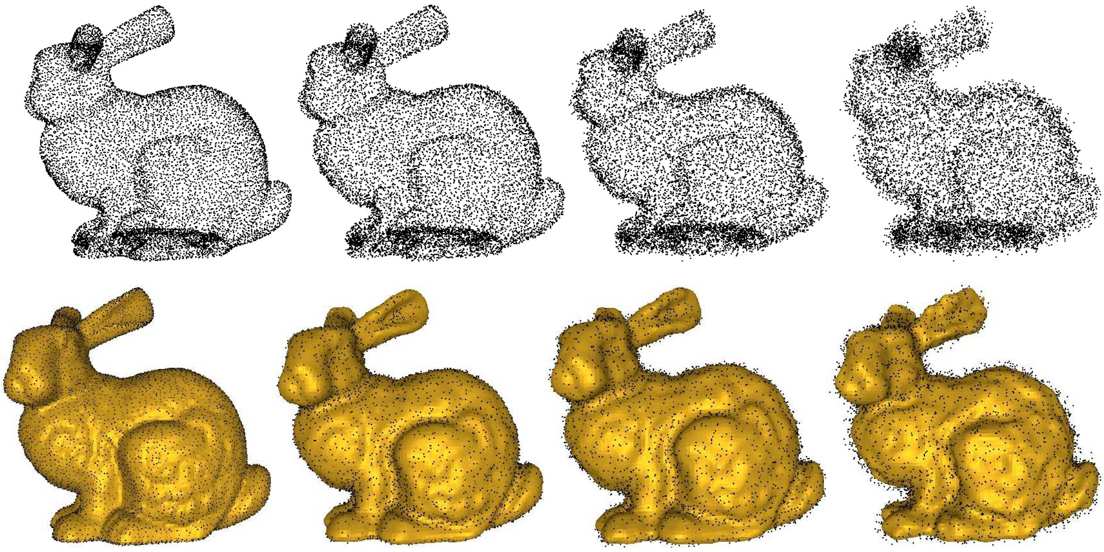
Model	#Points	Support size	Build	Solve	Total
Bunny	34835	0.005	2.453	8.078	
Chinese Dragon	152807	1.861	9.875	65.062	
Armadillo	165954	1.668	8.437	34.172	
Hand_olivier	195945	0.010	8.141	97.531	
Sforza	253419	4.557	10.500	118.047	

Figure 3 shows an example with part of the given samples having no normals. We substituted part of the normals with zero vectors randomly.

(a) (b) (c) (d)

Figure 3: Reconstruction of the Egea. From left to right, 0, 40, 60 and 80 percent of the given normals were substituted with zero vectors randomly.

Figure 4 illustrates an example with an amount of Gaussian noise added. The results show the resilience to noise in the data, with the parameter ρ proportionally to the standard deviation of the noise.



(a) (b) (c) (d)

Figure 4: Reconstruction of the Stanford bunny after adding Gaussian noise with standard deviation , from left to right, 0.1, 0.5, 1.0 and 2.0 percent of the radius of the smallest enclosing sphere, and parameter ρ of 100, 500, 1000 and 2000 respectively.

Additionally, we compared our method with the Poisson reconstruction from [3] to a two-torus model (4352 points) in Fig.5. We note that even at octree depth 10, the Poisson-based mesh is oversmoothed, mostly due to the interpolation of the normals of the octree leaves.

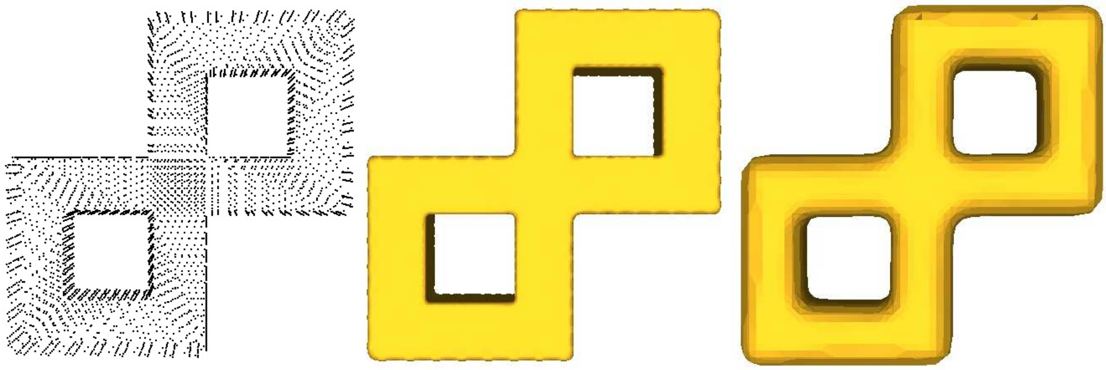


Figure 5: Comparison with the Poisson reconstruction. From left to right, the sample points, our method and Poisson reconstruction.

The main problem associated with compactly supported radial basis functions is that it often yields unwanted artifacts in addition to the lack of extrapolation across holes, as shown in Fig.2 and Fig.6. This is due to the finite extent of the compactly supported radial basis functions. Only those points within the radius of support of one of the centers have non-zero values. For all points outside this band, the implicit function values are zero, as shown in Fig.6(b).

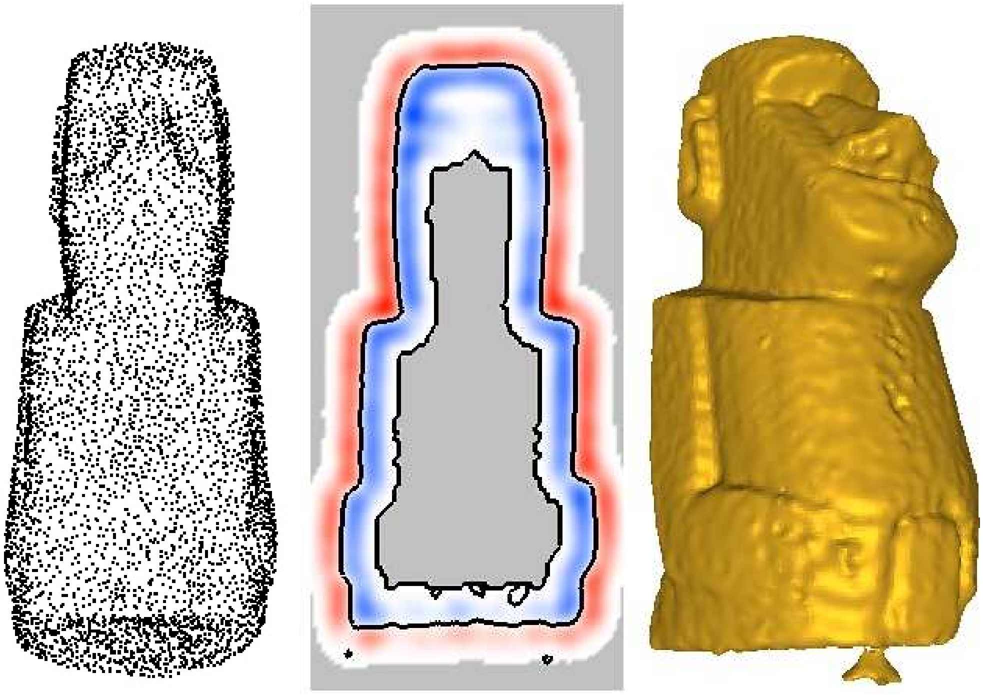


Figure 6: Reconstruction of the Moai model. Left: sample points; Middle: planar slice that cut the models. The colors represent the function values (red for positive value, blue for negative value), the black curve highlights its zero level and the gray color shows the undefined region; Right: the reconstruction with undesirable artifacts at the bottom.

5 Conclusion

We have presented an approach for implicit surface fitting with compactly supported radial basis functions by directly making use of surface normal vectors at sample points. The normal vectors are incorporated in a regularized regression problem that leads to a $n \times n$ positive definite linear system given n surface point/normal pairs. Compared with the widely used heuristic methods, our method avoids of introducing manufactured off-surface points and can fit much larger datasets effectively.

In future work, we wish to investigate how to avoid the unwanted artifacts and render these models effectively.

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